Log-Sobolev Inequality

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Log-Sobolev inequality is essential in proof of sampling algorithms. We first write out the Theorem.

Theorem 1. *Let $\psi \in \mathcal{C}^2(\mathbb{R}^n)$ and assume that there exists ρ ¿ 0 such that the linear operator $\mathbf{L}f = \Delta f - \nabla\psi \cdot \nabla f$ satisfies a Γ_2-criterium CD($\rho, +\infty$), then the probability measure μ_ψ satisfies a Poincare inequality*

$$Var_{\mu_\psi}(f) \leq \frac{1}{\rho}\int |\nabla f|^2 d\mu_\psi, \tag{1}$$

for all $f \in \mathcal{A}$ and a logarithmic Sobolev inequality

$$Ent_{\mu_\psi}(f) \leq \frac{1}{2\rho}\int \frac{|\nabla f|^2}{f} d\mu_\psi, \tag{2}$$

for all smooth and non-negative functions $f \in \mathcal{A}$.

An extended version of Proof for Poincare inequality [1].

Proof.

$$\begin{aligned}
\mathrm{Var}_{\mu_\psi}(f) &= \int f^2 d\mu_\psi - \left(\int f d\mu_\psi\right)^2 && (3)\\
&= \int (\mathbf{P_0} f)^2 d\mu_\psi - \int (\mathbf{P}_\infty)^2 d\mu_\psi && (4)\\
&= -\int (\mathbf{P_t} f)^2 d\mu_\psi \Big|_\infty^0 && (5)\\
&= -\int_0^\infty \frac{d}{dt}\int (\mathbf{P_t} f)^2 d\mu_\psi dt && (6)\\
&= -2\int_0^\infty \frac{d}{dt}\int (\mathbf{P_t} f)' (\mathbf{P_t} f) d\mu_\psi dt && (7)\\
(Prop.2.4) &= -2\int_0^{+\infty}\int \mathbf{L}\mathbf{P_t} f \mathbf{P_t} f d\mu_\psi dt && (8)\\
(Symmetric) &= -2\int_0^{+\infty}\int \mathbf{P_t} f \mathbf{L}\mathbf{P_t} f d\mu_\psi dt && (9)
\end{aligned}$$

For $\int \mathbf{L}(\mathbf{P_t} f)^2 d\mu_\psi$,

$$\begin{aligned}
\int \mathbf{L}(\mathbf{P_t} f)^2 d\mu_\psi &= \int \frac{d}{ds}\mathbf{P}_s(\mathbf{P_t} f)^2 d\mu_\psi && (10)\\
(Invariant) &= \frac{d}{ds}\int (\mathbf{P_t} f)^2 d\mu_\psi && (11)\\
&= 0. && (12)
\end{aligned}$$

Therefore,

$$\int_0^{+\infty}\int 2\mathbf{P_t}f\mathbf{L}\mathbf{P_t}f d\mu_\psi = \int_0^{+\infty}\int (2\mathbf{P_t}f\mathbf{L}\mathbf{P_t}f - 0)d\mu_\psi \tag{13}$$

$$= \int_0^{+\infty}\int (2\mathbf{P_t}f\mathbf{L}\mathbf{P_t}f - \mathbf{L}(\mathbf{P_t}f)^2)d\mu_\psi \tag{14}$$

$$(\text{Definition of } \Gamma) = -2\int_0^{+\infty}\int \Gamma(\mathbf{P_t}f, \mathbf{P_t}f)d\mu_\psi \tag{15}$$

$$= -2\int_0^{+\infty}\int \Gamma(\mathbf{P_t}f)d\mu_\psi \tag{16}$$

Therefore,

$$Var_{\mu_\psi}(f) = \int_0^{\infty} 2\int \Gamma(\mathbf{P_t}f)d\mu_\psi dt. \tag{17}$$

Now, we consider the term,

$$\Phi(t) = 2\int \Gamma(\mathbf{P_t}f)d\mu_\psi.$$

The time derivative of Phi is

$$\Phi'(t) = \left[2\int \Gamma(\mathbf{P_t}f)d\mu_\psi\right]' \tag{18}$$

$$= 2\int (2\mathbf{L}\mathbf{P_t}f\mathbf{L}\mathbf{P_t}f = 2\mathbf{L}\mathbf{P_t}f\mathbf{L}\mathbf{P_t}f - 2\mathbf{P_t}f\mathbf{L}\mathbf{L}\mathbf{P_t}f)d\mu_\psi \tag{19}$$

$$= 4\int \Gamma(\mathbf{P_t}f, \mathbf{L}\mathbf{P_t}f)d\mu_\psi \tag{20}$$

$$(\text{Using the same trick}) = 2\int (2\Gamma(\mathbf{P_t}f, \mathbf{L}\mathbf{P_t}f) - \mathbf{L}(\Gamma(\mathbf{P_t}f)))d\mu_\psi \tag{21}$$

$$= -4\int \Gamma_2(\mathbf{P_t}f)d\mu_\psi. \tag{22}$$

We have assumption on Γ_2 where

$$\Gamma_2(f) \geq \rho\Gamma(f).$$

We can see this Γ_2-criterium implies

$$\Phi'(t) \leq -2\rho\Phi(t).$$

By Grönwall's inequality,

$$\Phi(t) \leq e^{-2\rho t}\Phi(0).$$

This implies,

$$Var_{\mu_\psi}(f) = \int_0^{\infty} 2\int \Gamma(\mathbf{P_t}f)d\mu_\psi dt \tag{23}$$

$$= \int_0^{\infty} \Phi(t)dt \tag{24}$$

$$\leq \int_0^{\infty} e^{-2\rho t}\Phi(0)dt \tag{25}$$

$$\leq \int_0^{\infty} e^{-2\rho t}2\int \Gamma(\mathbf{P}_0 f)d\mu_\psi dt \tag{26}$$

$$(\mathbf{P}_0 f = f) \leq \int_0^{\infty} e^{-2\rho t}dt 2\int \Gamma(f)d\mu_\psi \tag{27}$$

$$\leq \frac{1}{\rho}\int \Gamma(f)d\mu_\psi dt. \tag{28}$$

□

Then, we show a proof of Log-Sobolev inequality based on **Logarithmic Sobolev inequality for diffusion semigroups** by *Ivan Gentil*.

Proof.

$$Ent_{\mu_\psi}(f) = \int f \log\left(\frac{f}{\int f d\mu_\psi}\right) d\mu_\psi. \tag{29}$$

We see

$$\begin{aligned}
\left[\int \mathbf{P_t} f \log \mathbf{P_t} f d\mu_\psi\right]_0^\infty &= \int \mathbf{P}_\infty f \log \mathbf{P}_\infty f d\mu_\psi - \int \mathbf{P}_0 f \log \mathbf{P}_0 f d\mu_\psi &(30)\\
&= \int \left(\int f d\mu_\psi\right) \log\left(\int f d\mu_\psi\right) d\mu_\psi - \int f \log f d\mu_\psi &(31)\\
&= -\int f \log f d\mu_\psi. &(32)
\end{aligned}$$

$$\begin{aligned}
Ent_{\mu_\psi}(f) &= -\int_0^\infty \frac{d}{dt}\int \mathbf{P_t} f \log \mathbf{P_t} f d\mu_\psi dt &(33)\\
&= -\int_0^\infty \int \mathbf{L}\mathbf{P_t} f \log \mathbf{P_t} f d\mu_\psi dt &(34)
\end{aligned}$$

Since $\mathbf{L}$ is symmetric and by the fact that

$$\mathbf{L}\varphi(f) = \varphi'(f)\mathbf{L}f + \varphi''(f)\Gamma(f)$$

We have

$$\begin{aligned}
\int \mathbf{L}\mathbf{P_t} f \log \mathbf{P_t} f d\mu_\psi &= \int \mathbf{P_t} f \mathbf{L} \log \mathbf{P_t} f d\mu_\psi &(35)\\
&= \int \mathbf{P_t} f \left(\frac{1}{\mathbf{P_t} f}\mathbf{L}\mathbf{P_t} f - \frac{1}{(\mathbf{P_t} f)^2}(\Gamma(\mathbf{P_t} f))\right) d\mu_\psi &(36)\\
&= \int \mathbf{L}\mathbf{P_t} f - \frac{1}{\mathbf{P_t} f}(\Gamma(\mathbf{P_t} f)) d\mu_\psi &(37)\\
&= -\int \frac{\Gamma(\mathbf{P_t} f)}{\mathbf{P_t} f} d\mu_\psi &(38)\\
(\Gamma(\log f) = \frac{1}{f^2}\Gamma(f)) &= -\int \Gamma(\log \mathbf{P_t} f)\mathbf{P_t} f d\mu_\psi &(39)
\end{aligned}$$

Then we have,

$$Ent_{\mu_\psi}(f) = \int_0^\infty \int \Gamma(\log \mathbf{P_t} f)\mathbf{P_t} f d\mu_\psi dt$$

Similarly as Poincase inequality proof,

$$\Phi(t) = \int \frac{\Gamma(\mathbf{P_t} f)}{\mathbf{P_t} f} d\mu_\psi.$$

Ignore the details here :), we take the derivative of $\Phi(t)$ then we will have

$$\Phi'(t) = -2\int \Gamma_2(\log \mathbf{P_t} f)\mathbf{P_t} f d\mu_\psi.$$

Once again, the Γ_2-criterium implies $\Phi'(t) \le -2\rho\Phi(t)$. Samely as

$$\Phi(t) \le e^{-2\rho t}\Phi(0).$$

Then we can see this inequality as

$$Ent_{\mu_\psi}(f) \leq \int_0^\infty e^{-2\rho t} dt \int \Gamma(\log f) f d\mu_\psi \tag{40}$$

$$= \frac{1}{2\rho} \int \Gamma(\log f) f d\mu_\psi \tag{41}$$

$$= \frac{1}{2\rho} \int \frac{|\nabla f|^2}{f} d\mu_\psi. \tag{42}$$

□

References

[1] Ivan Gentil. Logarithmic sobolev inequality for diffusion semigroups. *arXiv preprint arXiv:1009.3421*, 2010.